BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 6

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CONTENTS

I. REVIEW

Last time we:

(1) Calculated ramification points and indices in an example
(2) Defined the local normal form and degree of a morphism
(3) Started defining hyperelliptic curves

II. WHAT IS A BRANCH OF A MULTI-VALUED FUNCTION?

Let U be an open subset of $\mathbb{C}$ and suppose $f : U \to \mathbb{C}$ is a surjective, holomorphic function. Since f is surjective, then it has a right inverse, i.e., there exists a function $g : f(U) \to U$ such that $f \circ g = \mathrm{id}_{f(U)}$. There are many such g: to define one, for each $w \in f(U)$, simply choose some $z \in f^{-1}(w)$ and set $g(w) = z$. However, choosing this z so haphazardly means that g is very unlikely to be continuous, or have any other nice properties. And really we want much more: we'd like g to be holomorphic.

But even if we try to define g carefully, it will almost always have points of discontinuity. For instance, consider $f(z) = z^3$ with $g(w) = \sqrt[3]{w}$. [Show visualization at `https://openprocessing.org/sketch/1083105`.] With the definition we just showed, g is discontinuous along the positive real axis. So we can define $\sqrt[3]{w}$ continuously on $\mathbb{C} \setminus [0, \infty)$, but not on any larger set.

So by restricting the domain of the right inverse g, we can obtain a continuous function. In general, given a domain $D \subseteq f(U)$, a branch of f^{-1} on D is a continuous function $g : D \to U$ such that $f \circ g = \mathrm{id}_D$. Given such a g, we can even say something about its differentiability.

Theorem 1. *Suppose $f : U \to \mathbb{C}$ is holomorphic and g is a branch of f^{-1} on a domain $D \subseteq f(U)$. Fix $z_0 \in D$ and let $w_0 = g(z_0)$. If $f'(w_0) \neq 0$, then g is differentiable at z_0 and $g'(z_0) = 1/f'(w_0)$.*

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Reexamining the visualization, notice if we go around the origin three times, we do return to the value that we started with. So if we take three copies of $\mathbb{C}$, cut each of them along the real axis and then glue them together along these cuts, we can define a global cube root function $g : \mathbb{C} \to X$, where X is the resulting surface. We can write X as $\{(w, z) \in \mathbb{C}^2 : z^3 = w\}$ and the function z on X "is" the cube root function g, now defined and holomorphic on all of X.

I think this is historically why Riemann defined Riemann surfaces. And more complicated multi-valued functions lead us to familiar examples: if we consider $g(x) = \sqrt{x^3 - 1}$, the corresponding Riemann surface is the affine elliptic curve $E : y^2 = x^3 - 1$.

III. Hyperelliptic curves

Given an affine elliptic curve $y^2 = x^3 + Ax + B$ living inside the affine plane $\mathbb{A}^2$, we can easily find its closure in $\mathbb{P}^2$ simply by homogenizing the defining polynomial. Let's try to generalize this to what are known as hyperelliptic curves. Let $C : y^2 = x^5 - 1$ be an affine plane curve; let's try to determine its closure in $\mathbb{P}^2$ the same way. Is the resulting projective curve smooth?

While there are methods to resolve singularities, a more natural construction is the following. To define hyperelliptic curves, we need a weighted variant of the projective plane, whose definition we sketch below.

Definition 2. Given $g \in \mathbb{Z}_{\geq 1}$ define the weighted projective plane

$$\mathbb{P}(1, g+1, 1) := \frac{\mathbb{C}^3 \setminus \{(0,0,0)\}}{\sim}$$

where $(X, Y, Z) \sim (\lambda X, \lambda^{g+1} Y, \lambda Z)$ for all $\lambda \in \mathbb{C}^\times$.

Remark 3. One can similarly define $\mathbb{P}(a, b, c)$, however there is some strange behavior $\gcd(a, b, c) \neq 1$. Note that $\mathbb{P}(1, 1, 1) = \mathbb{P}^2$.

Just as with the usual projective plane, we have distinguished affine opens U_0, U_1, U_2, where X, Y, and Z are nonzero, respectively. However, the weights come into play when defining the standard open sets. We define

$$U_0 \to \mathbb{A}^2$$

$$[X : Y : Z] = \left[1 : \frac{Y}{X^{g+1}} : \frac{Z}{X}\right] \mapsto \left(\frac{Y}{X^{g+1}}, \frac{Z}{X}\right)$$

$$U_2 \to \mathbb{A}^2$$

$$[X : Y : Z] = \left[\frac{X}{Z} : \frac{Y}{Z^{g+1}} : 1\right] \mapsto \left(\frac{X}{Z}, \frac{Y}{Z^{g+1}}\right).$$

Note the conspicuous absence of a map for U_1! One can define a map on U_1 similarly to the above, but it actually won't be an isomorphism with $\mathbb{A}^2$, but rather the quotient $\mathbb{A}^2/\mu_{g+1}$ of $\mathbb{A}^2$ by the cyclic group of $(g+1)^{\text{st}}$ roots of unity.

However, note that $U_0 \cup U_2$ covers all of $\mathbb{P}(1, g+1, 1)$ except for the single point $[0 : 1 : 0]$ where $X = Z = 0$. It turns out that this point will never lie on our models of hyperelliptic curves, so we can safely ignore it.

Definition 4. A hyperelliptic curve over $\mathbb{C}$ is a smooth plane curve in $\mathbb{P}(1, g+1, 1)$ given by an equation of the form

$$Y^2 + h(X,Z)Y = f(X,Z)$$

(called a Weierstrass equation) where $f, h \in \mathbb{C}[X,Z]$ are homogeneous of degree $2g+2$ and $g+1$, respectively.

Remark 5. Consider $F := Y^2 + h(X,Z)Y - f(X,Z) \in \mathbb{C}[X,Y,Z]$, if we assign X and Z weight 1 and Y weight $g+1$, then F is weighted homogeneous of degree $2g+2$.

Since $\mathbb{C}$ has characteristic 0, we can complete the square and obtain a short Weierstrass equation:

$$Y^2 = f(X,Z)\,.$$

Proposition 6. *Let $C : Y^2 = F(X,Z)$ be a hyperelliptic curve, so on the open subset U_2 where $Z \neq 0$, C is given by $y^2 = f(x)$, where $f(x) = F(x,1)$.*

(a) The map $\iota : (x,y) \mapsto (x,-y)$ extends to an involution (i.e., a morphism such that $\iota^2 = \mathrm{id}$) defined on all of C. (This is called the hyperelliptic involution.)

(b) The map

$$\pi : C \to \mathbb{P}^1$$
$$[X : Y : Z] \mapsto [X : Z]$$

is a degree 2 morphism that is ramified above the roots of f, and if f has odd degree, also at the point $[1:0:0]$.

Proof. We first consider π on U_2, where it is given by $(x,y) \mapsto x$, where $x = X/Z$ and $y = Y/Z^{g+1}$. Given $Q = x_0 \in \mathbb{A}^1$, then $\pi^{-1}(Q)$ consists of the points (x_0, y_0), where y_0 is a solution of the equation

$$y^2 = f(x_0)\,.$$

There are two such solutions, counted with multiplicity, so π has degree 2. By constancy of degree,

$$2 = \deg(\pi) = \sum_{P \in \pi^{-1}(Q)} e_P(\pi)$$

so the ramification values of π are exactly the x_0 such that there is only one solution y_0. This occurs exactly when $f(x_0) = 0$, i.e., x_0 is a root of f.

If f has odd degree, then the weighted homogenization F has a factor of Z. (For instance, if the affine equation for the curve is $y^2 = f(x)$ with $f(x) = x^5 - 1$, then the weighted homogenized equation is $Y^2 = X^5 Z - Z^6$.) Letting $Q = [1:0] = \pi([1:0:0])$, then we compute $\pi^{-1}(Q)$ by subsituting $X = 1$, $Z = 0$ into the equation for C, obtaining $Y^2 = 0$. Thus $\pi^{-1}(Q)$ consists of only one point, hence π is ramified at $[1:0:0]$. ☐

IV. Differentials

Some of the notation for defining differentials can be a bit cumbersome, so let's begin with an example to fix ideas.

Example 7. Let's define a differential on $\mathbb{P}^1$. Writing $[X_0 : X_1]$ for the homogeneous coordinates on $\mathbb{P}^1$, recall that we have a holomorphic atlas consisting of the open sets $U_0 = \{X_0 \neq 0\}$ and $U_1 = \{X_1 \neq 0\}$ with coordinate maps

$$\varphi_0 : U_0 \xrightarrow{\sim} \mathbb{C}$$
$$[X_0 : X_1] = [1 : X_1/X_0] \mapsto X_1/X_0$$

$$\varphi_1 : U_1 \xrightarrow{\sim} \mathbb{C}$$
$$[X_0 : X_1] = [X_0/X_1 : 1] \mapsto X_0/X_1\,.$$

Denote the coordinates on the images of φ_0 and φ_1 by z_0 and z_1, respectively. Consider the differential on dz_1 on $\mathrm{img}(\varphi_1) = \mathbb{C}$. Even if you don't know a rigorous definition for dz_1, you probably know what it is: something we can integrate. (People with background in differential topology will probably say something about covector fields, but it basically amounts to the same thing.) So we have a differential on one chart of $\mathbb{P}^1$: let's see if it extends to all of $\mathbb{P}^1$. Let's work heuristically first. On $U_0 \cap U_1$ we have $z_1 = 1/z_0$, so we should have

$$dz_1 = d(1/z_0) = -\frac{1}{z_0^2}dz_0$$

which gives us the expression for dz_1 or U_0. More rigorously, on $U_0 \cap U_1$ z_1 and z_0 are related by the transition function $\varphi_1 \circ \varphi_0^{-1}$. We have $z_1 = (\varphi_1 \circ \varphi_0^{-1})(z_0)$ which sends

$$z_0 \xmapsto{\varphi_0^{-1}} [1 : z_0] = [1/z_0 : 1] \xmapsto{\varphi_1} 1/z_0$$

so we find

$$dz_1 = (\varphi_1 \circ \varphi_0^{-1})'(z_0)\, dz_0\,.$$

Definition 8. Given charts $(U_i, \varphi_i), (U_j, \varphi_j)$ on a Riemann surface, and $P \in U_i \cap U_j$ denote the deriviate of their transition function at P by

$$\frac{dz_i}{dz_j}(P) := \left(\varphi_i \circ \varphi_j^{-1}\right)'(\varphi_j(P))\,.$$

Definition 9. A meromorphic differential (one-form) ω on a Riemann surface X consists of an open cover $\{U_i\}_i$ of X and a collection of meromorphic functions $\{f_i : U_i \to \mathbb{C}\}_i$ such that

$$f_j = f_i \frac{dz_i}{dz_j}$$

on $U_i \cap U_j$ for all i, j. If the f_i are holomorphic for all i, then ω is called holomorphic.

We denote the set of all meromorphic differentials on X by $\mathcal{M}^1(X)$, and the set of holomorphic differentials by $\Omega(X)$ or $\mathcal{O}^1(X)$.

Remark 10. We often write this $\omega|_{U_i} = f_i\, dz_i$ and express the compatibility condition by $f_i\, dz_i = f_j\, dz_j$.

Remark 11. For differential geometers, a differential is a section of the cotangent bundle. Our definition is really the same thing. What we've done is specify an invertible sheaf, which is often called a line bundle, by specifying its transition functions.

Definition 12.

- Let X be a Riemann surface with an atlas $\{U_i\}_i$ where the local coordinate on U_i is z_i. Given a meromorphic function $f \in \mathcal{M}(X)$, define
$$\frac{\partial f}{\partial z_i}(P) := (f \circ \varphi_i^{-1})'(\varphi_i(P)) .$$
- Given a meromorphic function $f \in \mathcal{M}(X)$, define the meromorphic differential df to be the collection $\left\{ \frac{\partial f}{\partial z_i} \right\}_i$. We often express this by writing $df|_{U_i} = \frac{\partial f}{\partial z_i}\, dz_i$.

Note that given a meromorphic differential ω on X and a meromorphic function $f \in \mathcal{M}(X)$, then $f\omega$ is again a meromorphic differential, so $\mathcal{M}^1(X)$ is a vector space over $\mathcal{M}(X)$.

Proposition 13. *Given meromorphic differentials $\omega_1, \omega_2 \in \mathcal{M}^1(X)$, then there exists a meromorphic function $h \in \mathcal{M}(X)$ such that $\omega_1 = h\,\omega_2$. Thus $\mathcal{M}^1(X)$ is 1-dimensional as a vector space over $\mathcal{M}(X)$.*

Proof. The idea is to define h as ω_1/ω_2. More precisely, given an atlas $\{U_i\}_i$ of X, then for each i we have $\omega_1 = f_i^1\, dz_i$ and $\omega_2 = f_i^2\, dz_i$ for some $f_i^1, f_i^2 \in \mathcal{M}(U_i)$. So given $P \in U_i$, define
$$h(P) := \frac{f_i^1(P)}{f_i^2(P)} .$$
It remains to show that this is well-defined. If U_j is another chart with $P \in U_j$, then $f_j^1 = f_i^1 \frac{dz_i}{dz_j}$ and $f_j^2 = f_i^2 \frac{dz_i}{dz_j}$, so
$$\frac{f_j^1(P)}{f_j^2(P)} = \frac{f_i^1(P)\frac{dz_i}{dz_j}(P)}{f_i^2(P)\frac{dz_i}{dz_j}(P)} = \frac{f_i^1(P)}{f_i^2(P)} .$$
Thus the definition of $h(P)$ is independent of the choice of chart, so h is well-defined. □

Definition 14. Let $\omega \in \mathcal{M}^1(X)$ be a meromorphic differential on X and write $\omega = f_i\, dz_i$ with respect to some holomorphic atlas $\{U_i\}_i$. A point $P \in X$ is a zero or pole of ω if it is a zero or pole of f_i, where $P \in U_i$. In this case we define the order of vanishing of ω at P as $\operatorname{ord}_P(\omega) := \operatorname{ord}_P(f_i)$.

Example 15 ($\mathbb{P}^1$ has no nonzero holomorphic differentials)**.** As we have seen, letting $z = X_1/X_0$, the differential dz has a double pole at infinity. We can use this observation to show that there are no holomorphic differentials of $\mathbb{P}^1$. By the above, any differential ω can be written as $\omega = f(z)\, dz$ for some $f \in \mathcal{M}(\mathbb{P}^1)$. Recall that every nonconstant meromorphic function has at least one pole. In order for $f(z)\, dz$ to be holomorphic everywhere, then f must be constant. But in order for $f(z)\, dz$ to be holomorphic at ∞, f must also have a zero of order ≥ 2 at ∞. The only way this can occur is if f is the constant zero function, so $\omega = 0$.